SELF-ENTRAINMENT OF A POPULATION OF
COUPLED NON-LINEAR OSCILLATORS

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Temporal organization of matter is a widespread phenomenon over a macroscopic world in far from thermodynamic equilibrium. A previous study on chemical instability[1] implies that a simplest nontrivial model for a temporally organized system may be represented by a macroscopic self-sustained oscillator Q obeying the equation of motion

$$\dot{Q} = (i\omega + \alpha)Q - \beta|Q|^2Q \ ,$$

$$\alpha, \beta > 0 .$$

(1)

Consider a population of such oscillators Q_1, Q_2,$\cdots Q_N$ with various frequencies, and introduce interactions between every pair as follows.

$$\dot{Q}_s = (i\omega_s + \alpha)Q_s + \sum_{r \neq s} v_{rs}Q_r - \beta|Q_s|^2Q_s \ ,$$

$$r, s = 1, 2, \cdots N .$$

(2)

We found that it is possible to construct from (2) a soluble model for a community exhibiting mutual synchronization or self-entrainment above a certain threshold value of the coupling strength. Such a type of phase transition has been considered by Winfree[2] without resorting to specialized models but only phenomenologically.

Our simplifying assumptions are:
(I) $v_{rs} = v/N$ independently of r and s,
(II) $\alpha, \beta \to \infty$ but α/β, ω_s, v = finite,
(III) $N \to \infty$.

Let us put $Q_s = \rho_s e^{i\varphi_s}$. Owing to the assumption (II), the amplitude ρ_s may be fixed at $\sqrt{\alpha/\beta}$. Thus we have only to consider the equation

$$\dot{\varphi}_s = \omega_s + \frac{v}{N} \sum_r \sin(\varphi_r - \varphi_s) \ .$$

(3)

As an illustration, we summarize the results obtained when the distribution of the native frequency is a Lorentzian with the peak at ω_0 and the width γ. In this case the threshold condition is

$$\eta \equiv 2|\gamma/v| = 1 .$$

(4)

For $\eta < 1$ the asymptotic behavior of $\varphi_s(t)$ as $t \to \infty$ has two possibilities depending on the value of its native frequency ω_s. Thus we classify the oscillators into two groups :

(A) $\left| \dfrac{\omega_s - \omega_0}{v\sqrt{1 - \eta}} \right| < 1$.

The oscillators satisfying this condition are mutually synchro-
nized and one finds

$$\varphi_s(t) = \widetilde{\omega}_s t + \psi_s + \psi' , \tag{5}$$

$$\widetilde{\omega}_s = \omega_0 , \tag{6}$$

where ψ_s is a single-valued function of ω_s and ψ' is an arbitrary
constant but should be the same for all s belonging to this group.

(B) $\left| \dfrac{\omega_s - \omega_0}{v\sqrt{1 - \eta}} \right| > 1$.

The oscillators of this group fail to synchronize, but their fre-
quencies are shifted. We find

$$\varphi_s(t) = \widetilde{\omega}_s t + f_s(t), \tag{7}$$

where

$$\widetilde{\omega}_s = \omega_0 + (\omega_s - \omega_0)\sqrt{1 - \frac{v^2(1 - \eta)}{(\omega_s - \omega_0)^2}} \tag{8}$$

and

$$f_s(t) = f_s\left(t + \frac{2\pi}{\widetilde{\omega}_s}\right) . \tag{9}$$

If we define an order parameter σ by the number of the os-
cillators belonging to (A) divided by N, then we find

$$\sigma = \begin{cases} \dfrac{2}{\pi} \tan^{-1} \dfrac{2\sqrt{1 - \eta}}{\eta} & (\eta < 1) \\ 0 & (\eta > 1) . \end{cases} \tag{10}$$

Finally the stationary distribution of the effective frequency ω may
be expressed as

$$f(\widetilde{\omega}) = \sigma\delta(\widetilde{\omega} - \omega_0)$$
$$+ \frac{\gamma}{\pi} |\widetilde{\omega} - \omega_0| / [\{(\widetilde{\omega} - \omega_0)^2 + \gamma^2 + v^2\chi\}\sqrt{(\widetilde{\omega} - \omega_0)^2 + v^2\chi}] \tag{11}$$

where

$$\chi = \begin{cases} 1 - \eta & (\eta < 1) \\ 0 & (\eta > 1). \end{cases}$$

An infinitely sharp peak at $\widetilde{\omega} = \omega_0$ corresponds to a macroscopic os-
cillation. The background represents the contribution from the os-
cillators which have failed to synchronize with the main oscillation.
The intensity of the background near the center has been reduced
drastically due to the factor $|\widetilde{\omega} - \omega_0|$. It is very interesting to notice

that our spectrum resembles that of the α-rhythm of human brain wave.[3]

References
1] Y. Kuramoto and T. Tsuzuki, Prog. Theor. Phys. 52 (1974), 1399.
2] A.T. Winfree, J. Theor. Biol. 16 (1967), 15.
3] N. Wiener, "Cybernetics" MIT Press, Cambridge, Mass., 1948.